SM221 – Test #4 – Fall 2008 – Solutions

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1. Given the scalar field $f(x,y,z)$, which of the following produces a vector field?

   (a) $\nabla\times(\nabla f)$ (b) $\nabla\cdot(\nabla f)$ (c) $\nabla(\nabla\times f)$

   (d) all of the above   (e) none of the above

- (a) produces a vector field.
- (b) produces a scalar field.
- (c) is not meaningful.

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2. Find the potential (if it exists) for the vector field $\vec{F}=\langle 2xy+y^2, x^2+2xy\rangle$:

   (a) $2x$ (b) x^2y+xy^2 (c) $2x^2y^2$ (d) $2y$ (e) does not exist

- $\int(2xy+y^2)\,dx = x^2y+xy^2+h_1(y)+c$
- $\int(x^2+2xy)\,dy = x^2y+xy^2+h_2(x)+c$
- From this, one can conclude that the potential is x^2y+xy^2 (i.e. $h_1(y)=h_2(x)=c=0$)

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3. A particle moves through a vector field $\vec{F}=\langle 2y, 2x+4\rangle$ from the point (0,-5) to (-3,4) on the curve $y=x^2-5$. The line integral $\int_C \vec{F}\cdot d\vec{r}$ is:

   (a) $\langle 8,-2\rangle$ (b) $\langle -3,12\rangle$ (c) 0   (d) -36   (e) 12

- It can be shown that the vector field $\vec{F}$ is conservative, $\left(i.e.\ \frac{\partial}{\partial x}(2x+4)=2=\frac{\partial}{\partial y}(2y)\right)$.
- The potential of $\vec{F}$ is determined to be $f(x,y)=2xy+4y$.
- Therefore, by the fundamental theorem of vector calculus $\int_C \vec{F}\cdot d\vec{r} = f(x,y)\Big|_{(x_0,y_0)}^{(x_1,y_1)}$
- Thus $\int_C \vec{F}\cdot d\vec{r} = (2xy+4y)\Big|_{(0,-5)}^{(-3,4)} = -24+16-(0-20)=12$

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4. Given the vector field $\vec{F}=\langle 2xy+y^2, x^2+2xy\rangle$ and a path C that goes from (0,0) to (2,1), find $\int_C \vec{F}\cdot d\vec{r}$ (assume the path is linear):

   (a) $\langle 5, 8\rangle$ (b) $\sqrt{89}$ (c) 0   (d) 6   (e) $6\sqrt{2}$

- Since we showed that the vector field above has a potential $f(x,y)=x^2y+xy^2$
- $\int_C \vec{F}\cdot d\vec{r} = f(2,1)-f(0,0)=2^2(1)+2(1)^2=6$

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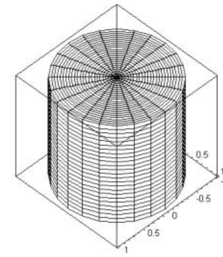
5. What is the flux through sides, top, and bottom of a cylinder with radius $r=1$ and height $h=2$ that sits inside the vector field $\vec{F}=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$ (note: the cylinder sits on the *xy*-plane)?



(a) 20π (b) 2π (c) 11π (d) 28π (e) 0

- Use Divergence Theorem: Flux= $\iint_S \vec{F}\cdot d\vec{S}=\iiint_E div\ \vec{F}\,dV$
- $div\ \vec{F}=\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(y^3)+\frac{\partial}{\partial z}(z^3)=3x^2+3y^2+3z^2$
- Therefore $\iiint_E div\ \vec{F}\,dV=\iiint_{Cylinder}(3x^2+3y^2+3z^2)\,dV=\int_0^{2\pi}\int_0^2\int_0^1(3r^2+3z^2)\,r\,drdzd\theta=11\pi$

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6. Let $f(x,y,z)=x^2+y^2+z^2$ be the potential for a force field $\vec{F}(x,y,z)$. Determine the following:

(a) $\vec{F}(x,y,z)=\nabla f=\langle 2x,2y,2z\rangle$

(b) $div\ \vec{F}(x,y,z)=\frac{\partial}{dx}\vec{F}+\frac{\partial}{dy}\vec{F}+\frac{\partial}{dz}\vec{F}=2+2+2=6$

(c) $curl\ \vec{F}(x,y,z)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{dx} & \frac{\partial}{dy} & \frac{\partial}{dz}\\ 2x & 2y & 2z\end{vmatrix}=0$

- Note we also know by theorem that $\nabla\times\nabla f=0$

(d) Circle the properties that describe $\vec{F}(x,y,z)$: conservative incompressible irrotational

- Conservative because $\vec{F}(x,y,z)=\nabla f$
- Not incompressible because $div\,\vec{F}\neq 0$
- Irrotational because $curl\,\vec{F}=0$

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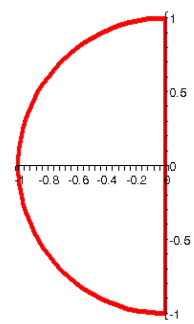
7. Compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y)=\langle y^2,-x^2\rangle$ and *C* is the positively oriented curve consisting of a line segment from (0,-1) to (0,1) and the left half of the circle $x^2+y^2=1$.



- Use Greens Theorem $\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$ where $Q=-x^2$ and $P=y^2$.
- Therefore $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-2x-2y$
- *dA* is the area described by the half circle. This region is best described in polar coordinates where $0\le r\le 1$ and $\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$.
- $\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\int_{\pi/2}^{3\pi/2}\int_0^1(-2x-2y)r\,drd\theta$.
- Now express *x* and *y* in polar coordinates … i.e. $x=r\cos\theta$ and $y=r\sin\theta$
- Therefore

$$\int_{\pi/2}^{3\pi/2}\int_0^1(-2x-2y)r\,drd\theta=-2\int_{\pi/2}^{3\pi/2}\int_0^1(r\cos\theta+r\sin\theta)r\,drd\theta=-\frac{2}{3}\int_{\pi/2}^{3\pi/2}(\cos\theta-\sin\theta)\,d\theta=\frac{4}{3}$$

- If you kept your integral in Cartesian coordinates you would have used:

$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}(-2x-2y)\,dxdy$ or $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(-2x-2y)\,dydx$ to get the answer

8. Consider the scalar field $f(x,y,z) = xyz$. Calculate the line integral along the curve $x^2 + y^2 = 4$ and z=3 going from the point (2,0,3) to (0,2,3).

- Use scalar form of line integral $\int_C f(x,y,z)ds = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt$ where $a \le t \le b$
- Parameterize the curve C: $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 3\rangle$ where $0 \le t \le \pi/2$.
- Therefore: $\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 0\rangle \rightarrow |\vec{r}'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = 2$
- $f(\vec{r}(t)) = 12\cos(t)\sin(t)$.
- $\int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt = 12\int_0^{\pi/2} \sin(t)\cos(t) = 6\sin^2(t)\Big|_0^{\pi/2} = 6$

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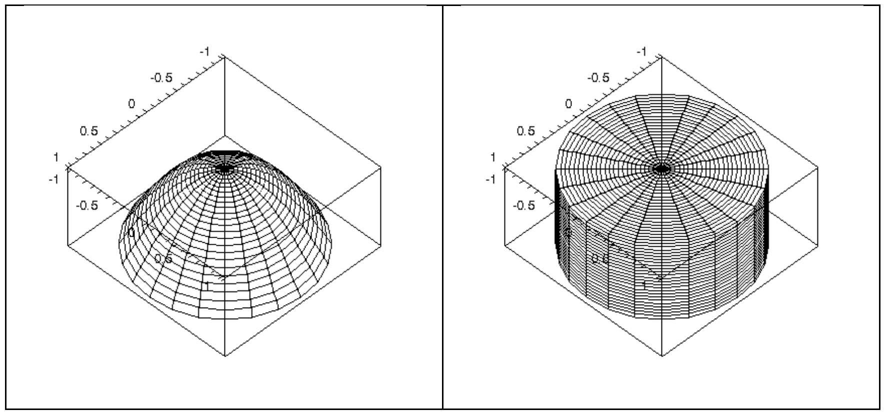
9. Find $\iint_S curl\ \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle -y, x, z\rangle$ and S is the paraboloid $z = 1 - x^2 - y^2$ where $0 \le z \le 1$ (note: the paraboloid is open on the *xy*-plane). How does this compare to $\iint_S curl\ \vec{F}\cdot d\vec{S}$ for a cylinder with radius $r = 1$ and height $h = 1$ which is closed at the top and open on the bottom?



- Use Stokes Theorem: $\iint_S curl\ \vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$.
- Here, C is the intersection between the paraboloid and the *xy*-plane, namely $x^2 + y^2 = 1$.
- Parameterize $C \rightarrow \vec{r}(t) = \langle \cos(t), \sin(t), 0\rangle$ where $0 \le t \le 2\pi$
- This implies $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$ and $F(r(t)) = \langle -\sin(t), \cos(t), 0\rangle$
- Therefore $F\cdot\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle\cdot\langle -\sin(t), \cos(t), 0\rangle = \sin^2(t) + \cos^2(t) = 1$
- $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} dt = 2\pi$
- **… or calculate** $\iint_S curl\ \vec{F}\cdot d\vec{S}$ directly …
- $curl\ \vec{F} = \langle 0,\ 0,\ 2\rangle$ and $d\vec{S} = \vec{N}dA$ where $\vec{N} = \langle 0,0,1\rangle$ (i.e. the normal to simplest surface enclosed the path … a circle in the xy-plane).
- Therefore $\iint_S curl\ \vec{F}\cdot d\vec{S} = 2\iint_D dA = 2\times$(area of the circle in the xy-plane)= $2(\pi r^2) = 2\pi$
- There is no change in $\iint_S curl\ \vec{F}\cdot d\vec{S}$ for the cylinder, since we use the same curve C to evaluate the cylinder.
