Sample Test 4 Solutions

SM 221 - VECTOR CALCULUS

TEST DATE — THURSDAY 12/2/2004

Sample Test 4 — Solutions

① When $x>0$ and $y>0$, the vector field points down (−) and to the right (+). Therefore $\frac{\partial f}{\partial x}>0$, $\frac{\partial f}{\partial y}<0$. This is only true for (e) where $\underline{\underline{\nabla f = \langle \frac{x}{4}, -\frac{y}{4} \rangle}}$

$\underline{\underline{\text{(e)}}}$

② $$W = \int_C \nabla f \cdot d\vec{r} = f(\underset{\nearrow \atop (0,2)}{\vec{r}(b)}) - f(\underset{\uparrow \atop (3,4)}{r(a)})$$

$$\Rightarrow (0+2^2)^{-1/2} - (3^2+4^2)^{-1/2} = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

$\underline{\underline{\text{(b)}}}$

③ Recall Greens Theorem

$$\int_C P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{f(x,y)} dA$$

$$\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2) = 2x - 2y$$

$\underline{\underline{\text{(b)}}}$

(4) Recall Divergence Theorem : $\iint_S \vec{F} \cdot dS = \iiint_E \operatorname{div} \vec{F} \, dV$

$\vec{F} = \langle x^2, 0, 0 \rangle \quad \operatorname{div} \vec{F} = 2x$

$\Rightarrow \int_0^1 \int_0^1 \int_0^1 2x \, dx \, dy \, dz = (1)(1)(x^2)\Big|_0^1 = 1$

(e)

(5) $F(x,y,z) = \langle P, Q, 0 \rangle$

① $\nabla f = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \Rightarrow$ as we proceed up ward from $P(1,1) \Rightarrow Q$ appears to increase. $\therefore \frac{\partial Q}{\partial y} > 0$

② as we proceed to the right, P appears to increase

$\therefore \frac{\partial P}{\partial x} > 0$

$\Rightarrow \nabla f > 0 \qquad$ (a) or (b)

③ P appears to be dependent on x-only

Q appears to be dependent on y-only

$\therefore \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x) & Q(y) & 0 \end{vmatrix} = (0-0)i - (0-0)j + (0-0)k$

$= 0$

$\Rightarrow$ (a)

(6) (a) can't take "div" of a scalar field

(b) same as (a)

(c) $\nabla f \Rightarrow$ vector field; $\nabla \cdot \vec{F} \Rightarrow$ scalar field $\}$ can't add $\nabla f + \vec{\nabla} \cdot \vec{F}$

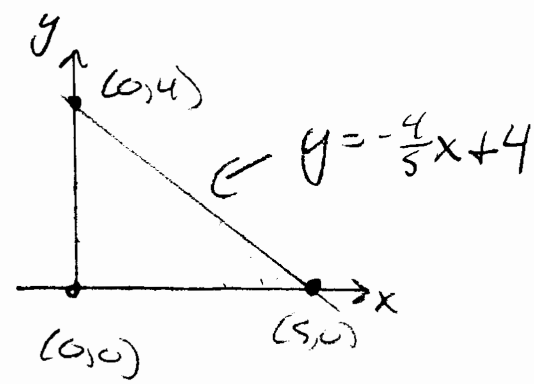
(d) $\nabla f \Rightarrow$ vector field; $\nabla \times \vec{F} \Rightarrow$ vector field $\}$ can add $\nabla f \times \vec{\nabla} \times \vec{F}$ ✓✓

(d)

(7) $\vec{\nabla} \times \vec{F} = 0$ for a conservative vector field

(e)

(8) $= \int \int_T (+1-3)\,dx\,dy$



$$= -2\int_0^5 \int_0^{-\frac{4}{5}x+4} dy\,dx$$

$$= -2\int_0^5 \left(-\frac{4}{5}x + 4\right) dx = -2\left(-\frac{2}{5}x^2 + 4x\right)\Big|_0^5$$

$$= -2(-10+20) = -20$$

(e)

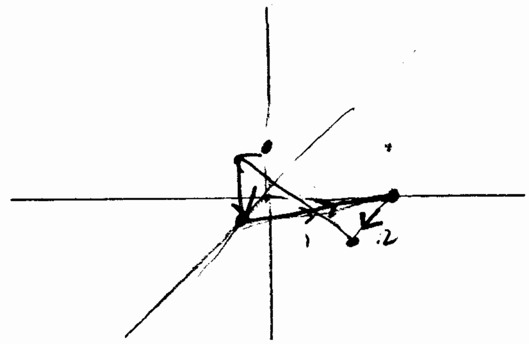
(9) $\vec{F} = \nabla f$

$\Rightarrow f = \sqrt{x^2+y^2+z^2}$

$\Rightarrow \left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right\rangle = \vec{F}$

(b)

⑩ If $\vec{F} = \langle y, -x, z \rangle$



C_1: $(1,0,0) \to (0,2,0)$

$\vec{r} = \langle 1-t, 2t, 0 \rangle$ $\quad \vec{F} = \langle 2t, t-1, 0 \rangle$

$d\vec{r} = \langle -1, 2, 0 \rangle$

$$\int_0^1 F \cdot dr = \int_0^1 (2t + 2t - 2)\, dt = -2$$

Note: all ~~parameterizations~~ parameterizations for C_1 thru C_4 are for $0 \le t \le 1$

C_2: $(0,2,0) \to (0,2,1)$

$\vec{r} = \langle 0, 2, t \rangle$ $\quad \vec{F} = \langle 2, 0, t \rangle$

$d\vec{r} = \langle 0, 0, 1 \rangle$

$$\int_0^1 F \cdot dr = \int_0^1 t\, dt = \frac{1}{2}$$

C_3: $(0,2,1) \to (1,0,1)$

$\vec{r} = \langle t, 2-2t, 1 \rangle$ $\quad \vec{F} = \langle 2-2t, -t, 1 \rangle$

$d\vec{r} = \langle 1, -2, 0 \rangle$

$$\int_0^1 F \cdot dr = \int_0^1 2 - 2t + 2t\, dt = 2\int_0^1 dt = 2$$

C_4: $(1,0,1) \to (1,0,0)$

$\vec{r} = \langle 1, 0, 1-t \rangle$ $\quad \vec{F} = \langle 0, -1, 1-t \rangle$

$d\vec{r} = \langle 0, 0, -1 \rangle$

$$\int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 t - 1\, dt = \tfrac{1}{2}t^2 - t \Big|_0^1 = -1/2$$

$C_1 + C_2 + C_3 + C_4 = 0$

a

(11) $\iint_S \vec{F} \cdot dS = \iiint \vec{\nabla} \cdot \vec{F}\, dV$

$\Rightarrow \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(2z)$

$= 2y - 2y + 2 = 2$

$\Rightarrow \iiint 2\, dV = 2 \boxed{\iiint dV} \leftarrow$ Volume of the sphere

radius of sphere

$= (2)\left(\frac{4}{3}\right)(\pi)(3)^3 = (2)(4)(9)\pi = \underline{72\pi}$

(d)

(12) $\frac{\partial}{\partial x} f(x,y) = 2xy \Rightarrow f = x^2 y + g_1(y) + k$

$\frac{\partial}{\partial y} f(x,y) = -x^2 \Rightarrow f = -x^2 y + g_2(x) + k$

$x^2 y = -x^2 y \Rightarrow$ can not resolve this conflict

(e)

7

???

(13) $\vec{\nabla}\cdot\vec{F} = \vec{F}_x + \vec{F}_y$

$\Rightarrow$ $\vec{F}$ appears to be independent of x $\Rightarrow$ $F_x = 0$

$\Rightarrow$ $|\vec{F}|$ increases as $|y|$ increases $\Rightarrow$ $F_y \neq 0$

$\therefore$ $\vec{\nabla}\cdot\vec{F} = \operatorname{div}\vec{F} \neq 0$ (c) or (e)

$\Rightarrow$ since $\vec{F}(x,y,z) \propto \langle ay, 0, 0\rangle$

$$\Rightarrow \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & 0 & 0 \end{vmatrix} = 0\hat{i} - 0\vec{j} - a\vec{k} = \vec{0}$$

$\therefore$ curl $= 0$ $\Rightarrow$

or ... there apear to be <u>no</u> rotational elements $\Rightarrow$ $\nabla\times\vec{F} = 0$

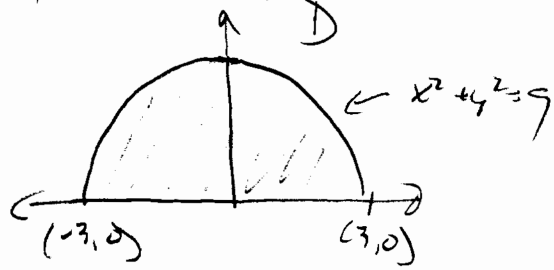
(C)

(14)

$$\oint_C y^3\,dx - x^3\,dy = \iint_D \frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\,dx\,dy$$

$$= \iint_D -3x^2 - 3y^2\,dx\,dy = -3\iint_D x^2+y^2\,dx\,dy$$

use polar: Domain →



$$= -3\int_0^{\pi}\int_0^{3} r^2\,r\,dr\,d\theta \qquad 0\le r\le 3,\quad 0\le\theta\le\pi$$

$$= -3\pi\int_0^3 r^3\,dr = -3\pi\left(\frac{1}{4}\right)r^4\Big|_0^3 = -\frac{243}{4}\pi \approx -191$$

(15) (a)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\sin y & y\cos x & xyz \end{vmatrix}$$

$$= \vec{i}\,(xz-0) - \vec{j}\,(yz) + \vec{k}\,(-y\sin x - x\cos y)$$

$$= \langle xz, -yz, -y\sin(x) - x\cos(y)\rangle$$

(b)

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x\sin(y)) + \frac{\partial}{\partial y}(y\cos(x)) + \frac{\partial}{\partial z}(xyz)$$

$$= \sin(y)+\cos(x)+xy$$

(16) $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \vec{\nabla} \cdot \vec{F} \, dV$

$\Rightarrow \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

$\Rightarrow \iiint_E 3 \, dV = 3 \iiint_E dV$

Volume of sphere with $r=1$

$= 3\left(\frac{4}{3}\right)(\pi)(1)^3 = \underline{\underline{4\pi}}$

(17) (a) $\nabla f = \langle 2x + 2xy, \; 2y + x^2 \rangle$

(b) $\vec{\nabla} \cdot \nabla f = \frac{\partial}{\partial x}(2x + 2xy) + \frac{\partial}{\partial y}(2y + x^2)$

$= 2 + 2y + 2 = \underline{\underline{4 + 2y}}$

(c) $\vec{\nabla} \cdot \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+2xy & 2y+x^2 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + (2x - 2x)\vec{k}$

$= \underline{\underline{0}}$

(18) $C_1: (0,0) \to (3,0) \Rightarrow \begin{matrix} x = 3t \\ y = 0 \end{matrix} \Big\} \; 0 \le t \le 1$

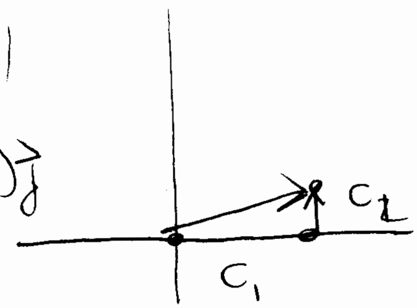
(a) $\vec{F}(x(t), y(t)) = \vec{F}(3t, 0) = 0i + (9t^2)j$

$\Rightarrow \vec{F} = \langle 0, 9t^2 \rangle$

$\Rightarrow \vec{r} = \langle 3t, 0 \rangle \Rightarrow d\vec{r} = \langle 3, 0 \rangle$

$\Rightarrow \vec{F} \cdot d\vec{r} = 0 \Rightarrow \int_0^1 0 \, dt = 0$

(b) $C_2 = (3,0) \to (3,1) \Rightarrow \begin{matrix} x = 3 \\ y = t \end{matrix} \Big\} \; 0 \le t \le 1$



$\vec{F}(x(t), y(t)) = \vec{F}(3, t) = (3t - t^2)\vec{i} + (9 + 2t)\vec{j}$

$\vec{r} = \langle 3, t \rangle \quad d\vec{r} = \langle 0, 1 \rangle$

$\Rightarrow \int \vec{F} \cdot d\vec{r} = \int_0^1 9 + 2t \, dt = 9t + t^2 \Big|_0^1 = 10$

(c) If I go from $(0,0) \to (3,1)$

then $\begin{matrix} x = 3t \\ y = t \end{matrix} \quad 0 \le t \le 1 \Rightarrow \vec{r}(t) = \langle 3t, t \rangle \quad d\vec{r} = \langle 3, 1 \rangle$

$F(x(t), y(t)) = (3t^2 - t^2)\vec{i} + (9t^2 + 2t)j$

$\Rightarrow F \cdot d\vec{r} = 6t^2 + 9t^2 + 2t = 15t^2 + 2t$

$\Rightarrow \int_0^1 15t^2 + 2t \, dt = 5t^3 + t^2 \Big|_0^1 = 6$

If $\vec{F}$ is conservative then

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}$$

$\Rightarrow$ but $\Rightarrow 10 \ne 6 \Rightarrow$ therefor $\vec{F}$ is not conservative

(19) (a) $\omega = \int_C \vec{F}\cdot d\vec{r} \Rightarrow$ $d\vec{r} = \langle 2t, 3t^2, 4t^3 \rangle$

$F(x(t), y(t), z(t)) = \langle t^6, t^5, t^7 \rangle$

$F\cdot dr = 2t^7 + 3t^7 + 4t^{10} = 5t^7 + 4t^{10}$

$\Rightarrow \int_0^1 5t^7 + 4t^{10}\, dt = \frac{5}{8}t^8 + \frac{4}{11}t^{11}\Big|_0^1 = \frac{5}{8} + \frac{4}{11} = \frac{55+32}{88} = \frac{87}{88}$

(b) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & yz \end{vmatrix} = z\vec{i} + x\vec{j} + y\vec{k} \neq 0$

$\therefore$ $\vec{F}$ is not conservative $\therefore$ ω will not be the same for a different path.

(20) $\iint_S \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{n}\,dA$ (can't use divergence theorem since surface not "closed")

$\Rightarrow$ plane: $-2x - 3y + z = 0 \Rightarrow \vec{n} = \langle -2, -3, 1 \rangle$
normal to plane (Ch. 9)

$\Rightarrow$ Rewrite $\vec{F}$ using $z = 2x + 3y$

$\Rightarrow \vec{F} = x(2x+3y)\vec{i} + xy\vec{j} + y(2x+3y)\vec{k}$

$\Rightarrow \vec{F}\cdot n = -2(2x^2 + 3xy) + (-3)(xy) + 2xy + 3y^2$
$= -4x^2 - 6xy - 3xy + 2xy + 3y^2 = -4x^2 - 7xy + 3y^2$

$\Rightarrow \iint \vec{F}\cdot\vec{n}\,dA = \int_0^1\int_0^2 -4x^2 + 7xy + 3y^2\, dx\, dy = -47/3$

by calc

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