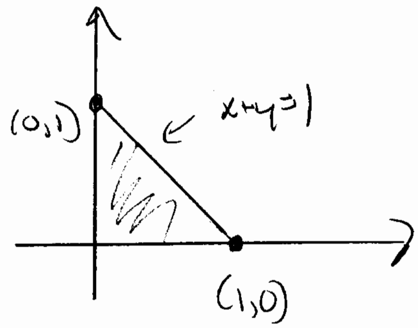
# Sample Test 3 - Multiple Choice

① R:




$0 \le x \le 1$
$0 \le y \le 1-x$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_0^{1-x} (x^2+2y^2)\,dy\,dx$$

$$\Rightarrow \int_0^1 \left(x^2 y + \tfrac{2}{3}y^3\right)\Big|_0^{1-x} dx = \int_0^1 \left(x^2 - x^3 + \tfrac{2}{3} - 2x + 2x^2 - \tfrac{2}{3}x^3\right) dx$$

$$= \int_0^1 \left(\tfrac{2}{3} - 2x + 3x^2 - \tfrac{5}{3}x^3\right) dx = \tfrac{2}{3}x - x^2 + x^3 - \tfrac{5}{12}x^4 \Big|_0^1$$

$$= \tfrac{2}{3} \cancel{-1+1} - \tfrac{5}{12} = \tfrac{8}{12} - \tfrac{5}{12} = \tfrac{3}{12} = 1/4$$

(a)

② Intersection is

$$x^2+y^2 = 1-x^2-y^2 \Rightarrow 2(x^2+y^2) = 1 \Rightarrow x^2+y^2 = 1/2$$

$$\Rightarrow \iint_D \left(1-x^2-y^2-(x^2+y^2)\right) dA = \iint_D (1-2x^2-2y^2)\,dA$$

$\Rightarrow$ convert to polar $\Rightarrow$ $0 \le \theta \le 2\pi$, $0 \le r \le (1/2)^{1/2}$

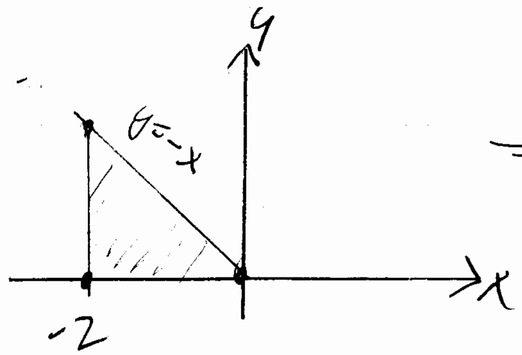
$dA = r\,dr\,d\theta$, $1-2x^2-2y^2 = 1-2r^2$

(c)

$$\Rightarrow \int_0^{2\pi}\int_0^{1/\sqrt{2}} (1-2r^2)\,r\,dr\,d\theta = (2\pi)\int_0^{1/\sqrt{2}} (r-2r^3)\,dr$$

$$= 2\pi\left(\tfrac{1}{2}r^2 - \tfrac{1}{2}r^4\right)\Big|_0^{1/\sqrt{2}} = 2\pi\left(\tfrac{1}{2}\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)\right) = 2\pi\left(\tfrac{1}{4} - \tfrac{1}{8}\right) = \frac{\pi}{4}$$

3)




$\Rightarrow$ to reverse order

$0 \le y \le 2$

$-2 \le x \le -y$

$$\Rightarrow \underline{\int_0^2 \int_{-2}^{-y} f(x,y)\,dx\,dy}$$

d

④ Surface Area $= \iint |u_x \times u_y|\, dx\,dy$

$u = \langle x, y, 2x+3y \rangle$

$u_x = \langle 1, 0, 2 \rangle \qquad u_y = \langle 0, 1, 3 \rangle$

$$u_x \times u_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2\vec{i} - 3\vec{j} + k$$

$$\Rightarrow |u_x \times u_y| = \sqrt{4+9+1} = \sqrt{14}$$

$$\Rightarrow SA = \int_0^2 \int_0^1 \sqrt{14}\, dy\, dx = (2)(1)\sqrt{14}$$

$$= 2\sqrt{14}$$

b

5) $\int_0^{\pi/2}\int_0^{\sin(x)} y\,dy\,dx$

$= \int_0^{\pi/2}\left(\frac{1}{2}y^2\right)\Big|_0^{\sin(x)} dx = \frac{1}{2}\int_0^{\pi/2}\sin^2 x\,dx$

$\Rightarrow$ use calculator

$\frac{1}{2}\int((\sin(x))^2, x, 0, \pi/2) =$

$\boxed{\pi/8}$

d

6) convert to polar

$\Rightarrow R := \{(r,\theta),\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ i.e a disk

$\Rightarrow dA = r\,dr\,d\theta$

$\Rightarrow f(x,y) = x^2 + y \Rightarrow f(r,\theta) = r^2\cos^2\theta + r\sin\theta$

$\Rightarrow \iint_R f(x,y)\,dA \Longrightarrow \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta + r\sin\theta)\,r\,dr\,d\theta$

or $\int_0^1\int_0^{2\pi}(r^2\cos^2\theta + r\sin\theta)\,r\,d\theta\,dr$

c

7)

xy plane $\Rightarrow z=0$

$y+z=1 \Rightarrow z=1-y \quad \therefore \quad 0 \le z \le 1-y$

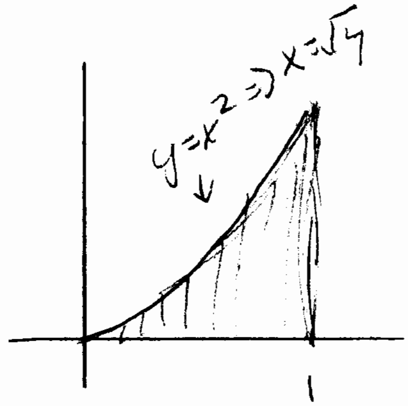
$\Rightarrow x^2+y^2=1 \Rightarrow y=\pm\sqrt{1-x^2}$

$\Rightarrow x = \quad \Rightarrow -1<x<1$

Therefore $\iiint_R z\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-y} z\,dz\,dy\,dx$

b//

(8)

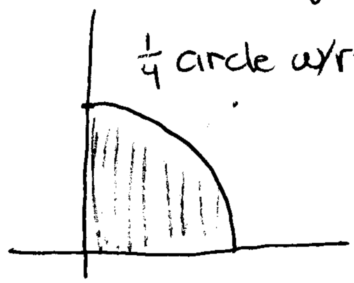



$\left.\begin{array}{l} 0\le x\le 1 \\ 0\le y\le x^2 \end{array}\right\} \Rightarrow \begin{array}{l} 0\le y\le 1 \\ \sqrt{y}\le x\le 1 \end{array}$

$\int_0^1\int_{\sqrt{y}}^1 \sin(y)\,dx\,dy$

a//

(9) Region: $0 \le y \le 3$, $0 \le x \le \sqrt{9-y^2}$



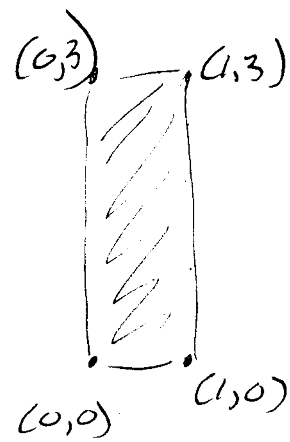

$\Rightarrow$ in polar

R: $0 \le r \le 3$, $0 \le \theta \le \frac{\pi}{2}$

$dxdy = r\,dr\,d\theta$ (in Polar)

$e^{x^2+y^2} = e^{r^2}$ (in Polar)

$\therefore \iint_R f(r,\theta)\,dA = \boxed{\int_0^{\pi/2}\int_0^3 e^{r^2}\, r\,dr\,d\theta}$




C

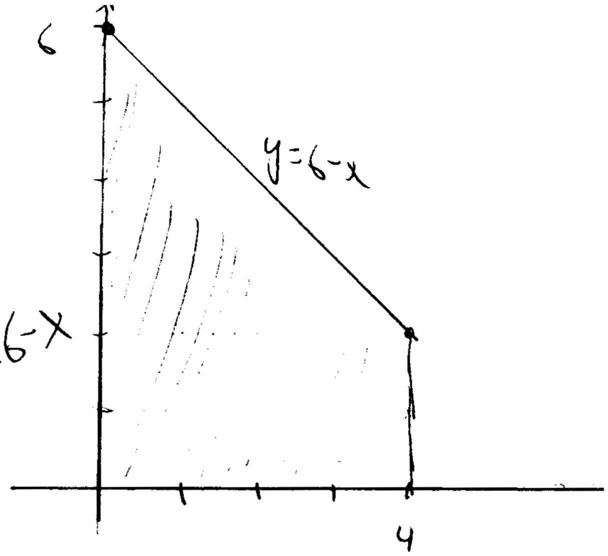
(10) $\int_0^1\int_0^3 223y^2\,dy\,dx$

$\Rightarrow = \int_0^1 \frac{223}{3} y^3 \Big|_0^3 dx$

$= \int_0^1 2007\,dx = \underline{2007}$

e

11)



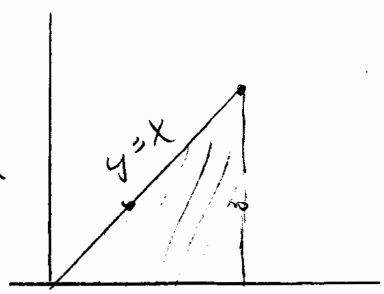

$0 \le y \le 6-x$

$0 \le x \le 4$

Trapezoid

c

12)




$0 \le y \le x$

$0 \le x \le 1$

if $0 \le y \le 1$
then $y \le x \le 1$

$\therefore$ Reverse order is

$$\int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

a

(B) Convert to Spherical Coordinates

$dA = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$\Rightarrow$ for a solid unit sphere

$0 \le \rho \le 1, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$

$\Rightarrow f(x,y,z) = x^2+y^2+z^2 \Rightarrow f(\rho,\theta,\phi) = \rho^2$

Integral becomes

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

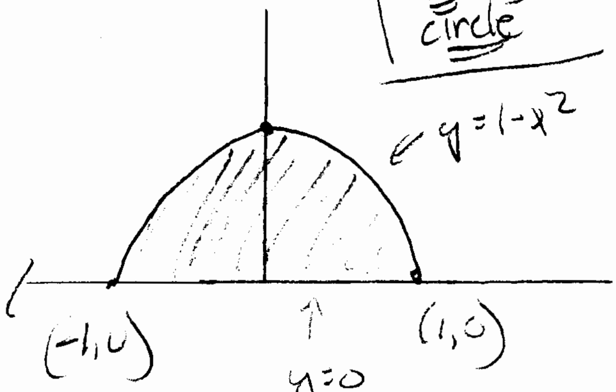
$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{5}\rho^5\Big|_0^1\right) \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = \frac{1}{5}(2\pi)\left(-\cos(\phi)\right)\Big|_0^{\pi}$$

$$= \frac{2}{5}\pi\left(-(-1)+1\right) = \frac{4}{5}\pi$$

C

(14) Region $\boxed{\text{Note this is not a circle}}$ → Don't use Polar




$-1 \le x \le 1$

$0 \le y \le (1-x^2)$

$0 \le z \le x^2+y^2$

$$m = \iiint_E \rho(x,y,z)\, dA$$

$$m = \int_{-1}^{1}\int_0^{(1-x^2)}\int_0^{(x^2+y^2)} z\, dz\,dy\,dx$$

a

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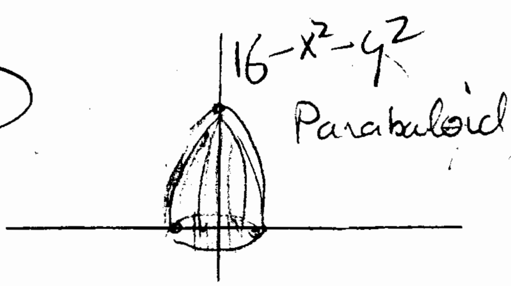
(15)




Region on xy-plane where $z=0$ is

$x^2+y^2=16$ (i.e. circle w/radius 4)

Solve in Cylindrical

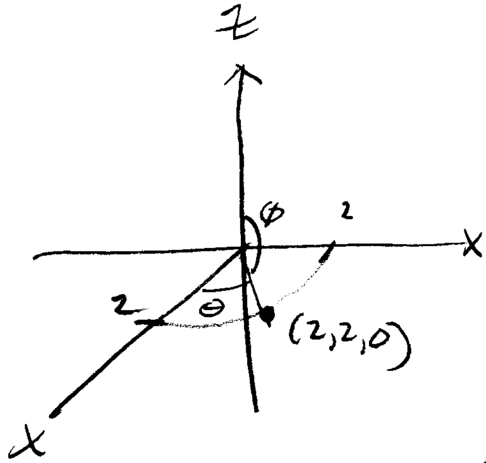
$$\int_0^{2\pi}\int_0^4\int_0^{16-r^2} r\,dr\,dz\,d\theta = 2\pi \int_0^4 (rz)\Big|_0^{16-r^2} d\theta$$

$$= 2\pi \int_0^4 16r - r^3\, dr = 2\pi\left(8r^2 - \tfrac{1}{4}r^4\right)\Big|_0^4 = 2\pi(128-64)$$

$= \underline{\underline{128\pi}}$

d

#16




$r = (x^2+y^2+z^2)^{1/2} = 2\sqrt{2}$

$\theta = \pi/4$

$\phi = \pi/2$

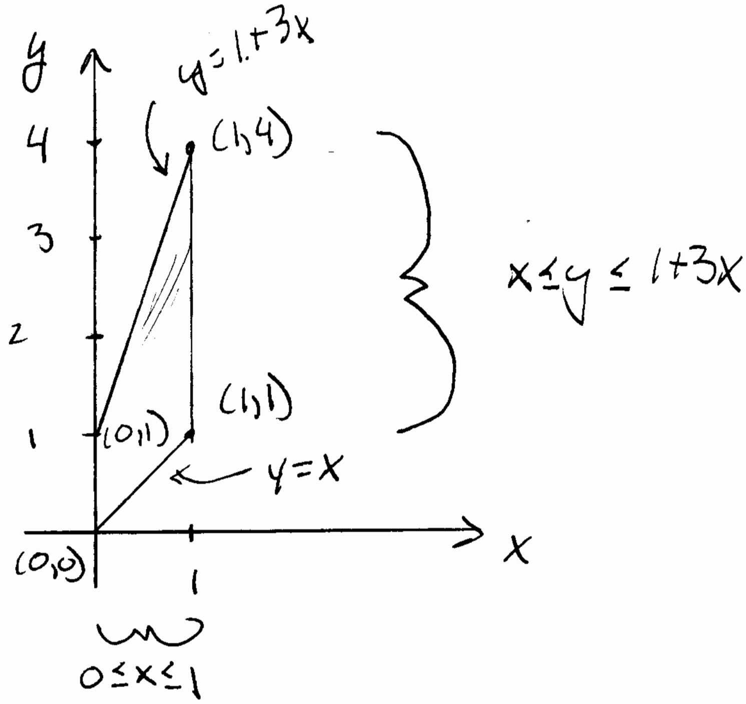
$\Rightarrow \left(2\sqrt{2}, \pi/4, \pi/2\right)$

$= \left(\sqrt{8}, \pi/4, \pi/2\right)$

a//

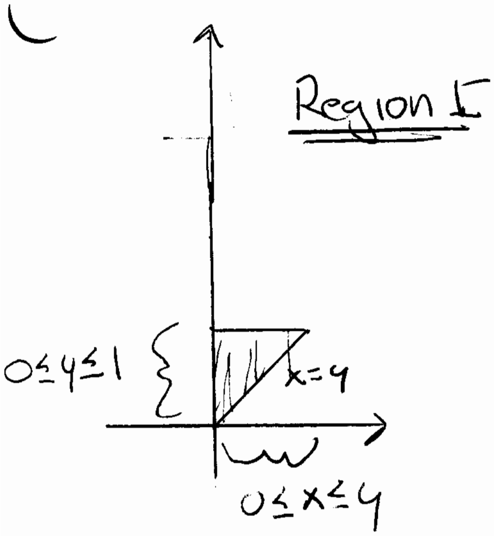
# FREE RESPONSE

① (a) $\displaystyle\int_0^1 \int_x^{1+3x} xy\,dy\,dx$




(b)

To Reverse the order of Integration we must break the region up into 2 Regions

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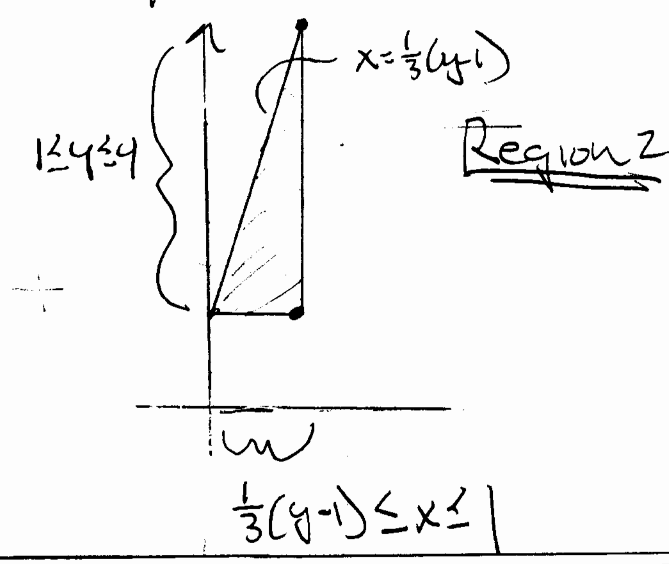
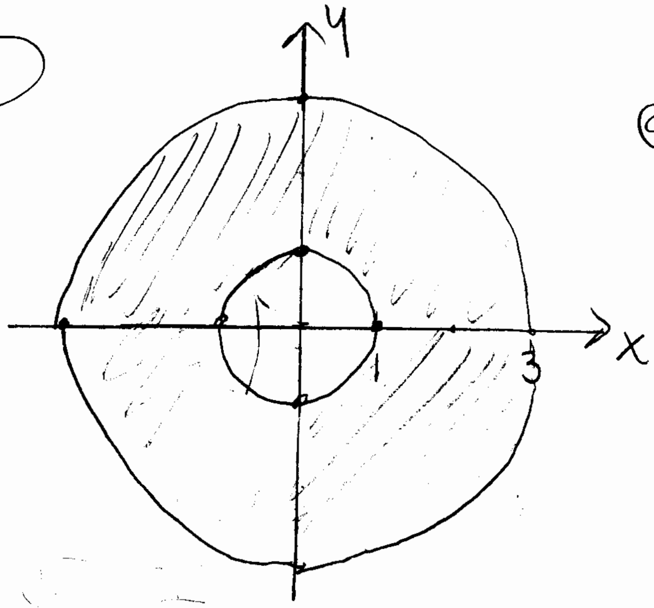



$$\boxed{\int_0^1 \int_0^y xy\,dx\,dy \quad + \quad \int_1^4 \int_{\frac{1}{3}(y-1)}^{1} xy\,dx\,dy}$$

Sum of Two Integrals!!

Do not evaluate!! ⇒ See attached Maple worksheet!

(2)




(a) Circular nature of these plots suggest that using polar coordinates would be advantageous!

i.e. $1 \le r \le 9 \quad 0 \le \theta \le 2\pi$

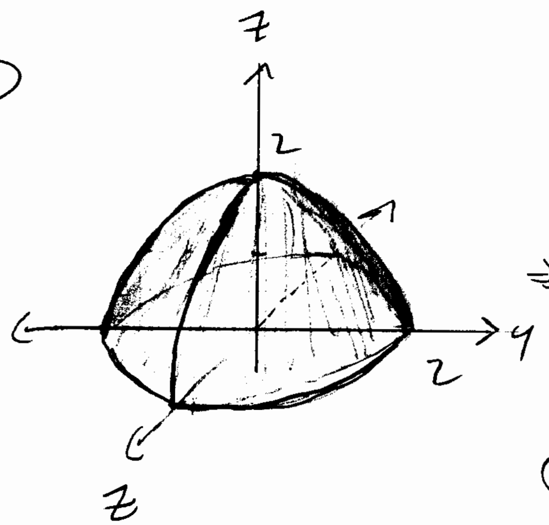
(b) $\underset{\leftarrow\, r^2}{} \quad \underset{\leftarrow\, r\,dr\,d\theta}{}$

$$\iint_A e^{x^2+y^2}\,dA \Rightarrow \int_0^{2\pi}\int_1^3 re^{r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{1}{2}e^{r^2}\Big|_1^3\right)d\theta = \int_0^{2\pi}\tfrac{1}{2}(e^9-e)\,d\theta$$

$$= \tfrac{1}{2}(2\pi)(e^9-e) = \underline{\underline{\pi(e^9-e)}}$$

(3) (a)




Hemisphere w/ Radius = 2

(b) $x^2+y^2 \Rightarrow r^2$

$dx\,dy \Rightarrow r\,dr\,d\theta$

$0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$$\Rightarrow \underline{\underline{\int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} (r^2+z^2)\,r\,dz\,dr\,d\theta}}$$

(c) $x^2+y^2+z^2 \Rightarrow \rho^2$

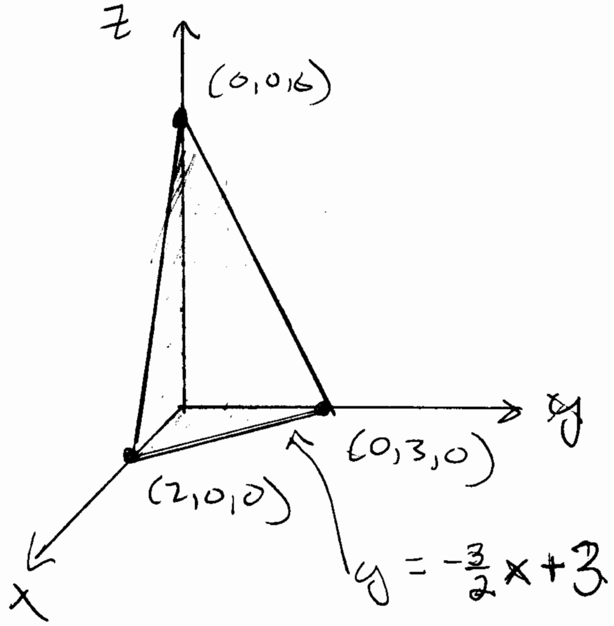
$dx\,dy\,dz \Rightarrow \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$0 \le \rho \le 2, \; 0 \le \phi \le \pi/2, \; 0 \le \theta \le 2\pi$

$$\Rightarrow \underline{\underline{\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (\rho^2)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}}$$

See Maple Attachment for Check

4




$$3x + 2y + z = 6$$

$$\left.\begin{array}{l} 0 \le z \le 6 - 2y - 3x \\ 0 \le y \le -\frac{3}{2}x + 3 \\ 0 \le x \le 2 \end{array}\right\} \text{limits of integratia}$$

$$V = \int_0^1 \int_0^{-\frac{3}{2}x+3} \left[ \int_0^{6-2y-3x} dz \right] dy\, dx$$

$$= \int_0^2 \left[ \int_0^{-\frac{3}{2}x+3} 6 - 2y - 3x \quad dy \right] dx$$

$$= \int_0^2 6y - y^2 - 3xy \Big]_0^{-\frac{3}{2}x+3} dx$$

$$= \int_0^2 -9x + 18 - \left(\tfrac{9}{4}x^2 - 9x + 9\right) - \left(-\tfrac{9}{2}x^2 + 9x\right) dx$$

$$= \int_0^2 \cancel{-9x} + 18 - \tfrac{9}{4}x^2 \cancel{+9x} - 9 + \tfrac{9}{2}x^2 - 9x \quad dx$$

$$= \int_0^2 \left(9 + \tfrac{9}{4}x^2 - 9x\right) dx = 9x + \tfrac{3}{4}x^3 - \tfrac{9}{2}x^2 \Big|_0^2$$

$$= 18 + 6 - 18 = 6$$

(5)

Use Polar Coordinates!

$$1 \le r \le 2, \quad 0 \le \theta \le \pi$$

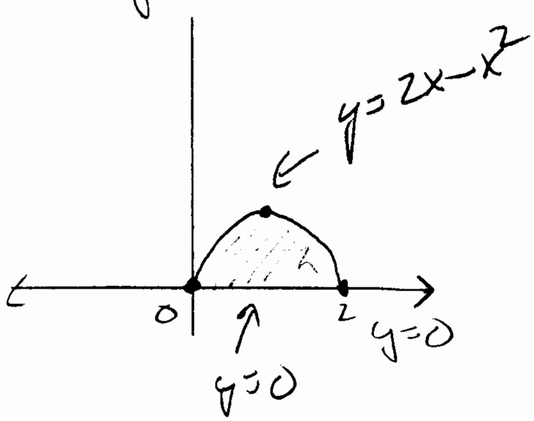
$$dA = r\,dr\,d\theta \qquad \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

$$\int_0^{\pi} \int_1^2 (r)(r)\,dr\,d\theta = \int_0^{\pi} \frac{1}{3} r^3 \Big|_1^2 \,d\theta$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right)(\pi) = \underline{\underline{\frac{7}{3}\pi}}$$

(6)

Region:




$$0 \le z \le x+y$$
$$0 \le y \le 2x - x^2$$
$$0 \le x \le 2$$

$$\int_0^2 \int_0^{2x-x^2} \int_0^{x+y} dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2x-x^2} (x+y)\,dy\,dx = \int_0^2 \left(xy + \frac{1}{2}y^2\right)\Big|_0^{2x-x^2} dy$$

$$= \int_0^2 2x^2 - x^3 + \frac{1}{2}\left(4x^2 - 4x^3 + x^4\right) dx = \int_0^2 \left(4x^2 - 3x^3 + \frac{1}{2}x^4\right) dx$$

$$= \frac{4}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{10}x^5 \Big|_0^2 = \frac{32}{3} - 12 + \frac{32}{10} = -\frac{4}{3} + \frac{16}{5}$$

$$= \frac{-20+48}{15} = 28/15$$

7 $\text{mass} = \iiint_R \rho \, dA$

$\Rightarrow$ Region is cylindrical

$0 \le r \le 5, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 3$

$\rho(x,y) = x^2 + y^2 \Rightarrow \rho(r,\theta) = r^2$

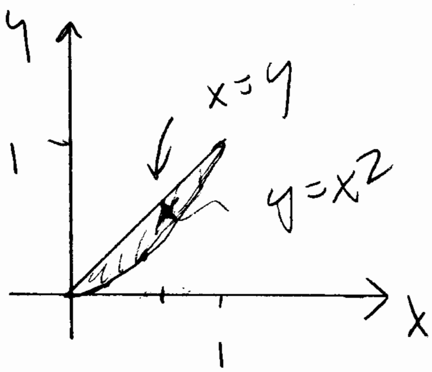
$dA = r\,dr\,d\theta\,dz$

$\Rightarrow \text{mass} = \int_0^{2\pi} \int_0^3 \int_0^5 r^2 \, r \, dr \, dz \, d\theta$

$= \int_0^{2\pi} \int_0^3 \frac{1}{4} r^4 \Big|_0^5 \, dz \, d\theta$

$= \frac{1}{\cancel{4}_2}(5)^4 (3)(\cancel{2}\pi) = \underline{\underline{\frac{1875}{2}\pi}}$

8)




$\rho(x,y) = 2y$

$\Rightarrow m = \int_0^1 \int_{x^2}^{x} (2y)\,dy\,dx = \int_0^1 y^2 \Big|_{x^2}^{x} dx$

$= \int_0^1 x^2 - x^4 \, dx = \frac{1}{3}x^3 - \frac{1}{5}x^5 \Big|_0^1 = \frac{1}{3} - \frac{1}{5}$

$= \frac{5-3}{15} = \frac{2}{15}$ //

$\Rightarrow M_x = \int_0^1 \int_{x^2}^{x} y(2y)\,dy\,dx = \int_0^1 \int_{x^2}^{x} 2y^2\,dy\,dx$

$= \int_0^1 \frac{2}{3}y^3 \Big|_{x^2}^{x} dx = \frac{2}{3}\int_0^1 x^3 - x^6 \, dx = \frac{2}{3}\left(\frac{1}{4}x^4 - \frac{1}{7}x^7\right)\Big|_0^1$

$= \frac{2}{3}\left(\frac{1}{4} - \frac{1}{7}\right) = \left(\frac{2}{3}\right)\left(\frac{3}{28}\right) = \frac{1}{14}$ //

$\Rightarrow M_y = \int_0^1 \int_{x^2}^{x} x(2y)\,dy\,dx = \int_0^1 xy^2 \Big|_{x^2}^{x} dx$

$= \int_0^1 (x^3 - x^5)\,dx = \frac{1}{4}x^4 - \frac{1}{6}x^6 \Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{2}{24} = \frac{1}{12}$ //

$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{1}{\cancel{12}_4} \times \frac{\cancel{15}^5}{2} = \frac{5}{8}$ //

$\bar{y} = \frac{M_x}{m} = \frac{1}{14} \frac{15}{2} = \frac{15}{28}$ //

9 Use the set up from problem 4

$$\text{mass} = \int_0^1 \int_0^{-3/2x+3} \int_0^{6-2y-3x} (x^2+z)\,dz\,dy\,dx$$

$\rho(x,y,z)$

$$\text{mass} = \frac{57}{5} \leftarrow \text{by maple}$$

**command for calculator**

$\int((\int((\int(x\wedge2+z, z, 0, 6-2y-3x), y, 0, -1.5x+3), x, 0, 2)$

$= \underline{\underline{11.4}}$

10) $z^2 = x^2 + y^2 \Rightarrow x^2+y^2 + x^2+y^2 = 8$

$\Rightarrow x^2+y^2 = 4$

Very Hard Problem!

intersection at $z=4$ or circle $x^2+y^2=4$

(a) For Cylindrical Coordinates

① $z^2 = x^2+y^2 \Rightarrow z^2 = r^2 \Rightarrow z=r$ bounded below

② $z^2 + x^2 + y^2 = 8 \Rightarrow z = (8-r^2)^{1/2}$ bounded above

③ intersection on circle $r^2=4 \Rightarrow r=2$

$\Rightarrow 0 \le r \le 2$

④ also $0 \le \theta \le 2\pi$

mass $\iiint_E \rho \, dV \Rightarrow \rho = z$ dV

$$\Rightarrow \int_0^{2\pi} \int_0^{2} \int_{r}^{(8-r^2)^{1/2}} z\,(r\,dz\,dr\,d\theta)$$

10 b for Sperical

recall

$x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\sin\phi$
$z = \rho\cos\phi$

also $\rho^2 = x^2+y^2+z^2$
$dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$\Rightarrow$ equation for sphere is $\rho^2 = 8 \Rightarrow \rho = \sqrt{8}$

equation for cone..

$\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)$

$\rho^2\cos^2\phi = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$

$\cancel{\rho^2}\cos^2\phi = \cancel{\rho^2}\sin^2\phi \Rightarrow \cos\phi = \sin\phi$

$\Rightarrow \tan\phi = 1 \Rightarrow \phi = \pi/4$

$\therefore\ 0 \le \rho \le \sqrt{8},\quad 0 < \phi < \pi/4,\quad 0 \le \theta \le 2\pi$

$\Rightarrow$ density $= z = \rho\cos\phi$

$$m = \iiint_E (\text{density})\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} (\rho\cos\phi)(\rho^2)\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^3\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta}$$

(10) c solve the spherical integral

note since $\sin\phi\cos\phi = \frac{1}{2}\sin 2\phi$

we can now write it as

$$\frac{1}{2}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^3 \sin 2\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\rho^4\Big|_0^{\sqrt{8}} \sin 2\phi \, d\phi \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/4} 16\sin 2\phi \, d\phi \, d\theta = 8\int_0^{2\pi}\left(-\frac{1}{2}\cos 2\phi\right)\Big|_0^{\pi/4} d\theta$$

$$= -4\int_0^{2\pi}(0-1)\, d\theta = 4(2\pi) = \underline{8\pi}$$