Sample Test Solutions

Test II

1) $\frac{\partial f}{\partial p}(2,10) = +3$

① "at T=20"

② "and P=10atm"

③ "the speed of sound is increasing"

④ "with increasing pressure"

⑤ "at a rate of $3\,\frac{m}{s}$"

b

2)
$$\begin{tikzcd} & b \arrow[dl] \arrow[dr] & \\ T \arrow[d, dash] & & C \arrow[d, dash] \\ t & & t \end{tikzcd}$$

$$\frac{db}{dt} = \frac{\partial b}{\partial T}\frac{dT}{dt} + \frac{\partial b}{\partial C}\frac{dC}{dt}$$

$$= (1000)(500) - 800(400)$$

$$= 500{,}000 - 320{,}000 = 180{,}000$$

e

3) $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2x+y,\ x \rangle$ "gradient"

$\Rightarrow \vec{\nabla} f(3,1) = \langle 7, 3 \rangle = 7\hat{i} + 3\hat{j}$ a

4) $f(x,y) = \sqrt{x^2+y^2} \Rightarrow z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2$

$\Rightarrow$ let z = constant, i.e. $z = r$

$\Rightarrow x^2+y^2 = r^2 \Longrightarrow$ family of circles! c

6) $\vec{V}_1 = \langle 1, 0, f_x \rangle = \langle 1, 0, 2e^{2x} \rangle$

$\vec{V}_2 = \langle 0, 1, f_y \rangle = \langle 0, 1, -\sin(x) \rangle$

$\vec{V}_1 (0,0,2) = \langle 1, 0, 2 \rangle$

$\vec{V}_2 (0,0,2) = \langle 0, 1, 0 \rangle$

$$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2i + k = \langle -2, 0, 1 \rangle$$

$\Rightarrow -2x + z = d$

+ plug in pt. $\langle 0,0,2 \rangle$

$\Rightarrow 2 = d \Rightarrow -2x + z = 2 \Rightarrow \underline{z = 2x + 2}$

a

7) $f_x = 2xy \Rightarrow f_{xy} = 2x \Rightarrow f_{xy}(2,3) = \underline{\underline{4}}$

d

(9) $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \langle 2x-2y, -2x, 2z \rangle$

$\vec{\nabla} f(1,1,2) = \langle 2(1)-2(1), -2(1), 2(2) \rangle$

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$= \langle 0, -2, 4 \rangle$

d

(10) $\vec{v} = \langle 4, 3 \rangle \Rightarrow$ implies $\vec{u} = \langle \frac{4}{5}, \frac{3}{5} \rangle$

"unit vector"

$\Rightarrow D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \langle 3, -2 \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle = \frac{12}{5} - \frac{6}{5}$

$= 6/5$

d

(11) $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u}$

$\Rightarrow \vec{\nabla} f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle$

$= \langle 5, -2 \rangle$

$\Rightarrow \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -3, 4 \rangle}{5} = \langle \frac{-3}{5}, \frac{4}{5} \rangle$

$\Rightarrow \vec{\nabla} f \cdot \vec{u} = \langle 5, -2 \rangle \cdot \langle \frac{-3}{5}, \frac{4}{5} \rangle = -23/5$

c

(12) $\vec{\nabla} f = \langle 2xy, x^2 \rangle$ "gradient vector"

$\Rightarrow \vec{\nabla} f(1,2,2) = \langle 2(1)(2), 1^2 \rangle = \langle 4, 1 \rangle$

$\Rightarrow$ rate of change $= |\vec{\nabla} f(1,2,2)|$

$= (4^2+1^2)^{1/2} = \sqrt{17}$

(e)

(13) $\int_0^{\pi} |r'(t)|\, dt = \int_0^{\pi} |\langle 2, \cos(t), -\sin(t)\rangle|\, dt$

$= \int_0^{\pi} (4+\cos^2(t)+\sin^2(t))^{1/2}\, dt = \int_0^{\pi} \sqrt{5}\, dt$

$= \sqrt{5}\, t \Big|_0^{\pi} = \sqrt{5}\,\pi \approx \underline{\underline{7.02}}$

(d)

(14) $\vec{v}_1 = \langle 1, 0, f_x \rangle = \langle 1, 0, 2e^{2x+y} \rangle$

$\vec{v}_2 = \langle 0, 1, f_y \rangle = \langle 0, 1, e^{2x+y} \rangle$

$\vec{v}_1(0,0,1) = \langle 1, 0, 2 \rangle$ $\vec{v}_2(0,0,1) = \langle 0, 1, 1 \rangle$

$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \langle -2, -1, 1 \rangle$

$\Rightarrow -2x - y + z = d \Rightarrow$ at pt $(0,0,1) \Rightarrow 1 = d$

$\Rightarrow -2x - y + z = 1$ or $z = 2x + y + 1$

(e)

(15) $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$

$\Rightarrow \vec{\nabla} f = \langle 2xy,\ x^2+z,\ y \rangle$

$\Rightarrow \vec{\nabla} f(3,-2,4) = \langle 2(3)(-2),\ 3^2+4,\ -2 \rangle$
$= \langle -12,\ 13,\ -2 \rangle$

$\Rightarrow \vec{\nabla} f \cdot \vec{u} = \langle -12, 13, -2 \rangle \cdot \langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle = 4/3$

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d

(17) $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} \Rightarrow \langle f_x, f_y \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = 1$

$$\Rightarrow \frac{3}{5} f_x + \frac{4}{5} f_y = 1$$

$$\Rightarrow 3f_x + 4f_y = 5$$

$D_{\vec{v}} f = \vec{\nabla} f \cdot \vec{v} \Rightarrow \langle f_x, f_y \rangle \cdot \left(\frac{5}{13}, \frac{12}{13} \right) = -1$

$$\Rightarrow \frac{5}{13} f_x + \frac{12}{13} f_x = -1$$

$$\Rightarrow 5f_x + 12 f_y = -13$$

$\rightarrow$ 2 equations w/ 2 unknowns

$$\begin{array}{lll} 5 \times & 3f_x + 4f_y = 5 & \Longrightarrow \\ 3 \times & 5f_x + 12f_y = -13 & \end{array} \qquad \begin{array}{rl} & 15f_x + 20f_y = 25 \\ - & 15f_x + 36f_y = -39 \\ \hline & -16f_y = 64 \end{array}$$

$$\Rightarrow \underline{f_y = -4}$$

$$\Rightarrow 3f_x + 4(-4) = 5$$

$$\Rightarrow 3f_x = 21 \Rightarrow \underline{\underline{f_x = 7}}$$

$$\Rightarrow \underline{\underline{\nabla f = \langle 7, -4 \rangle}}$$

(18) a) $P(100,25) = 8(100)^{1/2}(25)^{1/2}$
$= 8(10)(5) = \underline{\underline{400}}$

b) "... labor, production increases by "8,000 tons/month"

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c) $\frac{\partial P}{\partial K} = (8)\left(\frac{1}{2}K^{-1/2}\right)\left(L^{1/2}\right) = 4K^{-1/2}L^{1/2}$

$\frac{\partial P}{\partial K}(100,25) = \frac{4}{10}(5) = 2$

d) Find $\frac{\partial P}{\partial L} = (8)\left(K^{1/2}\right)\left(\frac{1}{2}L^{-1/2}\right)$

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$= 8(100)^{1/2}\left(\frac{1}{2}\right)\left(\frac{1}{(25)^{1/2}}\right)$
$= 8(10)\left(\frac{1}{2}\right)\left(\frac{1}{5}\right) = \underline{\underline{8}}$

$\Rightarrow \Delta P \approx \frac{\partial P}{\partial K}\Delta K + \frac{\partial P}{\partial L}\Delta L$

$= (2)(4) + 8(3) = \underline{\underline{32}}$ tons/month

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From part (c)

(19) $\vec{a} = \langle 0, 50\sin(\frac{\pi}{60}t) \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle c_1, c_2 - \frac{3000}{\pi}\cos(\frac{\pi}{60}t) \rangle$$

$$\vec{v}(0) = \langle c_1, c_2 - \frac{3000}{\pi} \rangle = \langle 100, 0 \rangle$$

$$\Rightarrow c_1 = 100, \quad c_2 = \frac{3000}{\pi}$$

$$\Rightarrow \vec{v}(t) = \langle 100, \frac{3000}{\pi}(1 - \cos\frac{\pi}{60}t) \rangle$$

$$\Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt = \langle 100t + c_1, \frac{3000}{\pi}t - \frac{180{,}000}{\pi^2}\sin(\frac{\pi}{60}t) + c_2 \rangle$$

$$\Rightarrow \vec{r}(0) = \langle c_1, c_2 \rangle = \langle 0, 0 \rangle$$

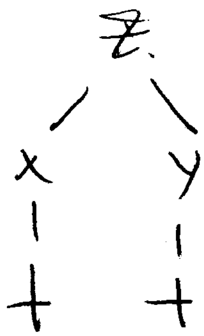
(a) $$\Rightarrow \vec{r}(t) = \langle 100t, \frac{3000}{\pi}t - \frac{180{,}000}{\pi^2}\sin(\frac{\pi}{60}t) \rangle$$

(b) $x = 20$ when $t = .2$

$$\Rightarrow y = \frac{3000}{\pi}(.2) - \frac{180000}{\pi^2}\left(\sin\frac{\pi}{60}(.2)\right)$$

$$\approx \underline{187.65} \text{ m}$$

(20)



$$\Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy)(2t) + (x^2)(1)$$

$\Rightarrow$ at $t=2$, $x=5$, $y=3$

$$\Rightarrow \left.\frac{dz}{dt}\right|_{t=2} = 2(5)(3)(2)(2) + 5^2(1)$$

$$= 120 + 25 = \underline{\underline{145}}$$

(21) a) $\frac{\partial D}{\partial T} \approx \frac{f(T+\Delta T, A) - f(T-\Delta T, A)}{2\Delta T}$

$\approx \frac{f(60, 2000) - f(30, 2000)}{60}$

$= \frac{1510 - 1110}{60} = 6\,^2\!/_3 \ \text{ft}/{}^\circ\text{F}$

b) $\frac{\partial D}{\partial A} \approx \frac{f(T, A+\Delta A) - f(T, A-\Delta A)}{2\Delta A}$

$\approx \frac{1530 - 1070}{4000} = \frac{460}{4000} = .115$

c) $\Delta D \approx \frac{\partial D}{\partial T}\underset{5^\circ}{\Delta T} + \frac{\partial D}{\partial A}\underset{500\text{ ft}}{\Delta A}$

$= (6\,^2\!/_3)(5) + (.115)(500) \approx 90.8$

$D(35, 2500) \approx 1280 + 90.8 = \underline{1370.8 \text{ ft}}$

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NOTE: your answers may vary based on how you approximate $\frac{\partial D}{\partial T}$ and $\frac{\partial D}{\partial A}$

(22) FIRST FIND CRITICAL POINTS

$f_x = 3x^2 - 12y = 0 \Rightarrow$

$f_y = -12x + 24y^2 = 0 \Rightarrow 12x = 24y^2 \Rightarrow x = 2y^2$

substitute into f_x equation

$\Rightarrow f_x = 3(2y^2)^2 - 12y = 0 \Rightarrow 3(4y^4) - 12y = 0$

$\Rightarrow 12y^4 - 12y = 0 \Rightarrow y^4 - y = 0$

$\Rightarrow y(y^3 - 1) = 0$

$\Rightarrow y = 0,\ y = 1$

$\therefore y = 0 \Rightarrow x = 2(0)^2 = 0$

$y = 1 \Rightarrow x = 2$

critical points at $(0,0)$, $(2,1)$

$f_{xx} = \frac{\partial}{\partial x}(3x^2) = 6x$

$f_{yy} = 48y$

$f_{xy} = -12$

at $(0,0) \Rightarrow f_{xx} = 0,\ f_{yy} = 0\ (f_{xy})^2 = 144$

$\Rightarrow D = f_{xx} f_{yy} - (f_{xy})^2 = -144 < 0$

$\therefore$ $\boxed{\text{saddle point @ } (0,0)}$ $\rightarrow$

at $(2,1)$

$f_{xx} = 6(2) = 12$ $\Rightarrow$ $D = f_{xx} f_{yy} - f_{xy}^2 = (12)(48) - 144$

$f_{yy} = 48(1) = 48$

$\Rightarrow D = 432 > 0$

Since $f_{xx} > 0$, $D > 0$ $\Rightarrow$ local min

$\therefore$ $\boxed{\text{local min @ } (2,1)}$

23) Find Critical Points

$f_x = 6x^2 + y^2 + 10x = 0$

$f_y = 2xy + 2y = 0 \Rightarrow y = 0$ $\leftarrow$ substitute in f_x

$\Rightarrow f_x = 6x^2 + 0^2 + 10x = 0 \Rightarrow 6x(x + \frac{5}{3})$

$\Rightarrow x = 0, -5/3$

$\Rightarrow$ critical points $(0,0)$, $(-5/3, 0)$

$f_{xx} = 12x + 10$

$f_{yy} = 2x + 2$

$f_{xy} = 2y$

at $(0,0) \Rightarrow f_{xx} = 10,\ f_{yy} = 2,\ f_{xy} = 0$

$D = (10)(2) - (0)^2 = 20 > 0$

$f_{xx}(x=0) = 10 > 0$

$\therefore$ $\boxed{\text{local minimum at } (0,0)}$

at $(-5/3, 0)$

$f_{xx} = -10 < 0 \quad f_{yy} = -\frac{10}{3} + 2 = -\frac{4}{3}$

$f_{xy} = 0 \Rightarrow D = (-10)(-4/3) - 0 > 0$

$f_{xx} < 0$

$\therefore$ $\boxed{\text{local max at } (-5/3, 0)}$